Harold J. Larson

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RELIABILITY GROWTH.

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by

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ABSTRACT:

A particular model is proposed for reliability growth and a prior distribution is assumed on the parameters. Various statements regarding the final reliability are then derived; a numerical example is included.

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1. INTRODUCTION

Modern DOD contracts frequently include incentive provisions for the reliability of the design used in building components of missiles, or a complete missile itself. Such an incentive provision commonly states that a target reliability (negotiated jointly by the government and the contractor) and a nominal reliability fee be paid for achievement of the target reliability. If the contractor can demonstrate a higher reliability than the target value, he will receive more than the nominal fee; if, on the other hand, his demonstrated reliability is less than the target value, he receives less than the nominal fee. Thus, the contractor has great incentive to demonstrate as high a reliability value as he possibly can.

Unfortunately, the individual items covered by such contracts (component parts or complete missiles) are generally very expensive, so a relatively small number N of items are funded for actual testing. In some instances, all N funded items may be tested simultaneously; failures that occur may then be analyzed, and the contractor may make design changes which are intended to remove the causes of the observed failures. However, if no further items are available to be tested, then the actual achieved reliability of the changed design cannot be estimated from test results, and yet the contractor would like to take advantage of any improvement in reliability which the change may have effected.

A specific model for describing this sort of situation is given by Corcoran, Weingarten, and Zehna [1]. They assume that N items are tested, each of which performs successfully (passes the test) or fails. Those items that fail can be classified as falling into one of K different failure modes. (Thus, we might think of the items as consisting of K components in logical series; the item passes if and only if all K components operate correctly, and will fail if any single one of the components does not operate correctly. Note that no two (or more) of these components can fail simultaneously.) It is assumed in [1] that the probability a_i of correcting the i^{th} failure mode (by the design change) is known, $i = 1, 2, \ldots, K$, for every failure mode that occurs in the N tests. It is pointed out in [1] that the achieved reliability (subsequent to all design changes) then is a random variable whose value is unknown (since the probabilities of success and of the occurrence of the various failure modes, for the original design, are not known quantities). The paper by Corcoran, Weingarten, and Zehna then goes on to define a quantity called mean reliability and to discuss, prior to any experimental results being available, ways of estimating mean reliability.

Larson [3] points out that any actual estimation of the achieved reliability (or more correctly, of some facet of its probability law) occurs only after the results of the N tests are available, and thus it would seem reasonable to refer only to the conditional distribution of the

achieved reliability, given the results of the N tests. This distribution is derived in [3]. He then speculates upon a method of calculating a lower bound for a confidence interval statement about the achieved reliability which does not require knowledge of the probabilities of success or of the occurrence of the various failure modes for the initial design. Earnest [2] mentions this speculated method and points out that it seems extremely difficult to apply in practise for values of interest for the bounding statement. Earnest then goes on to discuss a measure of reliability defined by Corcoran, Weingarten, and Zehna, and to discuss this quantity and its properties with an assumed prior distribution on the probabilities of success and the occurrence of the various failure modes.

It is the purpose of this paper to use the conditional distribution of achieved reliability derived in Larson [3] and a prior distribution for the parameters discussed by Earnest [2] to derive the (marginal) conditional distribution of achieved reliability, independent of the probabilities of success and of the occurrence of the various failure modes. This distribution can then be used to make any kind of probability statement of interest regarding the achieved reliability (conditioned on the outcomes of the N tests). Section 2 is devoted to a discussion of the assumed prior distribution and to a method of choosing the prior given information on the individual failure modes themselves. Section 3 then derives the (marginal) conditional distribution of achieved reliability, and section 4 presents a numerical example using synthetic data.

2. THE PARAMETERS AND THEIR PRIOR DISTRIBUTION

We shall adopt the notation used in both [1] and [3]. We assume that N items have been tested and that we observe N_0 successes and N_i failures in the i^{th} mode, $i = 1, 2, \ldots, K$. The initial reliability (for the initial design) is p_0, while the probability of occurrence of the i^{th} mode is q_i, $i = 1, 2, \ldots, K$. Thus, $N_0, N_1, \ldots, N_K$ constitute a sample of a multinomial random variable with parameters $N, p_0, q_1, \ldots, q_K$.

Those failure modes that occur in the N tests are identified and analyzed; a change is made in the initial design to attempt to remove some or all of these occurring failure modes. The probability that the i^{th} failure mode is corrected is a_i, a known quantity. Note that this model does not allow for any decrease in reliability to occur. Prior to the N tests the probability of the first failure mode occurring was q_1, while after the N tests (assuming this mode did occur) the probability of the first mode occurring is either still q_1 or 0, depending on whether the corrective action was successful. In any event, there is no chance of the probability of occurrence of this mode having been increased as a result of the change in design, within the framework of this model.

Let us examine the unknown parameters $p_0, q_1, \ldots, q_K$ a little more closely. As defined in [1] (and used in [2] and [3]), $p_0, q_1, \ldots, q_K$ are the probabilities of a single multinomial trial

falling into one of the $K+1$ classes: success, failure in i^{th} mode, $i = 1, 2, \ldots, K$; thus, $p_0 + \sum_{i=1}^{K} q_i = 1$. If, as was indicated in the introduction, the design concerned is for an item which consists of K subcomponents connected in series, so that the complete item will pass the test if and only if all K subcomponents operate correctly, then information collected on subcomponents can be used to get an indication of the value of p_0 and the q_i's ; and this information, in turn, can be used to construct a reasonable prior distribution of the parameters.

To illustrate the connection between the parameters $p_0, q_1, \ldots, q_K$ of the item contracted for and the probabilities that the individual subcomponents will work correctly, let p_i^* be the probability that the i^{th} subcomponent works correctly, $i = 1, 2, \ldots, K$; and thus $1 - p_i^*$ is the probability that this subcomponent does not work correctly. Then, if the subcomponents operate independently within the item, we have $p_0 = \prod_{i=1}^{K} p_i^*$. This does not exactly jibe with the assumption that no more than one failure mode can occur at a time, but does not represent a serious inconsistency for the uses envisaged for this model. The probability q_1 that the first failure mode occurs should increase and decrease as $1 - p_1^*$ increases and decreases; it would seem consistent to define

$$q_1 = \frac{1 - p_1^*}{K - \sum_{j=1}^{K} p_j^*} \left[1 - \prod_{j=1}^{K} p_j^* \right]$$

and, similarly, to define

$$q_i = \frac{1 - p_i^*}{K - \Sigma p_j^*} \left[1 - \Pi p_j^* \right] , \quad i = 2, 3, \ldots, K .$$

We then satisfy the requirement that

$$p_0 + \Sigma q_i = 1 ,$$

and we have defined q_i's that vary directly with their corresponding $(1 - p_i^*)$'s, as they should. Failure data on subcomponents could then be used to get estimates of $p_0, q_1, \ldots, q_K$ and to estimate parameters of the prior distribution of $p_0, q_1, \ldots, q_K$.

In using a Bayesian approach to solve a problem, it is generally necessary to act as though the parameters of the problem are themselves random variables. We then have a prior distribution of the parameters (measuring our degree of belief as to their values) which can be used in a variety of ways. In the context of the problem under discussion here, we shall assume that the probabilities of observing a success or of observing a failure in any one of the possible modes are random variables; we shall denote the random variables by capital letters and the particular values which they can take on by lower case letters. Thus, P_0 is the probability of observing a success when one of the items is tested, and Q_i is the probability of observing a failure in mode i, where $P_0 + \sum_1^K Q_i = 1$, as before.

An assumed prior distribution for $P_0, Q_1, \ldots, Q_K$ should satisfy several criteria. First, it should be such that it is capable of representing the assumed subjective distribution of $P_0, Q_1, \ldots, Q_K$. Second, it should be a distribution which is relatively easy to manipulate analytically (for the chosen problem) and not one which unnecessarily increases the mathematical analysis or the final computations in arriving at the desired solution. The parameters of the assumed prior should be capable of easy evaluation on the basis of previous information. In the context of the current problem, then, any chosen prior should be such that it is non-zero only over the possible ranges of P_0 and the Q_i's, namely on the simplex described by $P_0 \geq 0$, $Q_i \geq 0$, $i = 1, 2, \ldots, K$, $P_0 + \sum_{i=1}^{K} Q_i = 1$. Also, in the context of the current problem, $P_0, Q_1, \ldots, Q_K$ should not be independent random variables because of their structure.

Silva [5] (also reported in Earnest [2]) gives a form of the multivariate beta density function which would seem ideal for a prior distribution for $P_0, Q_1, \ldots, Q_K$. Defining $\underline{Q} = (Q_1, \ldots, Q_K)$, $\underline{q} = (q_1, \ldots, q_K)$, $p_0 = 1 - \sum_{i=1}^{K} q_i$, this distribution can be written

$$f_{\underline{Q}}(\underline{q}) = G\, p_0^{m_0 - 1} \prod_{i=1}^{K} q_i^{m_i - 1}$$

for $q_i \geq 0$, $p_0 \geq 0$, and $p_0 + \sum q_i = 1$, and

$$f_{\underline{Q}}(\underline{q}) = 0 ,$$

otherwise. The constant

$$G = \frac{\Gamma(m)}{\prod_{i=0}^{K} \Gamma(m_i)}$$

and $m = \sum_{i=0}^{K} m_i$, where $m_i > 0$, for all i. It can be shown that this prior is quite versatile, in the sense of being able to distribute the degrees of belief as to the values of $P_0, Q_1, \ldots, Q_K$ widely over the simplex (by changing the values of the m_i's) and, as we shall see, it is quite easy to use in the current context. It also includes dependence between $P_0, Q_1, \ldots, Q_K$, as was mentioned would be desirable. Silva shows that $E[Q_i] = \frac{m_i}{m}$, $i = 1, 2, \ldots, K$ (and thus $E[P_0] = \frac{m_0}{m}$), and then suggests that prior estimates or values of the mean values of the Q_i's can be used to determine the ratios $\frac{m_i}{m}$. This leaves only the parameter m to be specified (in order to completely determine the prior); Silva's suggestion at this point is to use as the value of m that number which minimizes the sum of squares of deviations between the prior estimates of the variances of $Q_1, Q_2, \ldots, Q_K$ and the variances imposed on these random variables by the form of the assumed prior. The value of m which achieves this minimization is

$$\frac{\sum_{i=1}^{K} [E(Q_i)]^2 [1 - E(Q_i)]^2}{\sum_{i=1}^{K} \hat{V}_i \, E(Q_i) [1 - E(Q_i)]} - 1 ,$$

where $\hat{V}_i$ is the prior variance of Q_i .

The foregoing discussion on the assignment of values to the parameters $m_1, m_2, \ldots, m_K, m$ has been presented because it is directly applicable to our reliability growth problem. To illustrate this, let us assume that the design we are considering is for an item which consists of K subcomponents connected in series. Let P_i^* be the true unknown probability that the i^{th} component will work correctly, $i = 1, 2, \ldots, K$, as before (except now we are thinking of these as random variables), and define $P_0 = \prod_{j=1}^{K} P_j^*$

$$Q_i = \frac{1 - P_i^*}{K - \Sigma P_j^*} \left[1 - \Pi P_j^* \right], \quad i = 1, 2, \ldots, K .$$

We assume that failure data is available for each of the K subcomponents. This failure data consists of r_i subcomponents of the given type having been tested under realistic conditions, and we know the number, X_i, of these that performed satisfactorily, $i = 1, 2, \ldots, K$. Then the maximum likelihood estimator of P_i^* (thought of as being a constant) is

$$\hat{P}_i^* = \frac{X_i}{r_i} , \quad i = 1, 2, \ldots, K ,$$

and the estimated variance of this estimator is

$$\hat{V}_i = \frac{\hat{P}_i^* (1 - \hat{P}_i^*)}{r_i} \quad ;$$

it is reasonable, in general, to assume that the various samples are independent. We can then define

$$\hat{Q}_i = \frac{1 - \hat{P}_i^*}{K - \Sigma \hat{P}_i^*} \left[1 - \Pi \hat{P}_i^* \right], \quad i = 1, 2, \ldots, K ,$$

and it seems reasonable to use this quantity as our a priori estimate of the mean of Q_i; thus, $\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_K$ will determine the ratios $\frac{m_1}{m}, \frac{m_2}{m}, \ldots, \frac{m_K}{m}$. For realistic values of $\hat{P}_i^*$ (say from .9 to 1), the quantity

$$\frac{1 - \Pi \hat{P}_i^*}{K - \Sigma P_i^*}$$

will not differ by a great deal from 1, and thus the variance of $\hat{Q}_i$ should not be much different than the variance of $1 - \hat{P}_i^*$; it should, in fact, be slightly smaller than that of $1 - \hat{P}_i^*$. Thus, since the variance of $1 - \hat{P}_i^*$ is the same as the variance of $\hat{P}_i^*$, we will be conservative in using $\hat{V}_i$ as the a priori variance of Q_i, $i = 1, 2, \ldots, K$. Then m is determined by Silva's least squares method to be

$$m = \frac{\Sigma [\hat{Q}_i]^2 [1 - \hat{Q}_i]^2}{\Sigma \hat{V}_i \hat{Q}_i [1 - \hat{Q}_i]} \quad ,$$

and we then have completely specified a reasonable prior for $P_0, Q_1, \ldots, Q_K$.

The next section gives the derivation of the marginal distribution of true reliability Z, after corrective action, conditional on the outcomes of the N items tested; it also displays the computation of a number b such that we are 95 percent certain that Z exceeds b, given the outcome of the tests (or any other level of confidence desired).

3. MARGINAL DISTRIBUTION OF Z

We are given that N items are to be tested; each item is either a success or, if a failure, falls into one of K possible distinct modes. The probability of success for a given item is P_0, and the probability that the item fails in mode i is Q_i, $i = 1, 2, \ldots, K$. The joint prior of $P_0, Q_1, \ldots, Q_K$ is the multivariate beta, $f_{\underline{Q}}(\underline{q})$, described in section 2. The number of successes observed (in the N tests) is N_0, and the number of failures observed in mode i is N_i, $i = 1, 2, \ldots, K$ (many of these N_i's may be zero). Assume that we observe $N_0 = n_0$, $N_1 = n_1, \ldots, N_s = n_s$ with $N_{s+i} = 0$, $i = 1, 2, \ldots, K - s$; the probability of this occurring, given values $P_0 = p_0$, $Q_i = q_i$, $i = 1, 2, \ldots, K$, is

$$p_{N|\underline{Q}}(\underline{n}|\underline{q}) = \frac{N!}{n_0!\, n_1! \ldots n_s!}\, p_0^{n_0} \prod_{i=1}^{s} q_i^{n_i} .$$

The posterior for $\underline{Q}$ then is

$$f_{\underline{Q}|N}(\underline{q}|\underline{n}) = \frac{\Gamma(N+m)}{\prod_{i=1}^{K} \Gamma(n_i + m_i)}\, p_0^{n_0 + m_0 - 1} \prod_{i=1}^{K} q_i^{n_i + m_i - 1}$$

where $n_{s+1} = n_{s+2} = \ldots = n_K = 0$. (See Earnest [2] for this derivation.)

Larson [3] shows that if we let A be the event $N_1 > 0$, $N_2 > 0$, $\ldots$, $N_s > 0$ and define Z to be the achieved reliability through the

design changes (a_i, the probability that the i^{th} failure mode is removed, is known), the conditional probability function of Z, given A, is as follows:

z	$P(Z = z \mid A)$
p_0	$\prod_{i=1}^{s} (1 - a_i)$
$p_0 + q_1$	$a_1 \prod_{i=2}^{s} (1 - a_i)$
$p_0 + q_2$	$a_2 (1 - a_1) \prod_{3}^{s} (1 - a_i)$
$\vdots$	$\vdots$
$p_0 + q_s$	$a_s \prod_{1}^{s-1} (1 - a_i)$
$p_0 + q_1 + q_2$	$a_1 a_2 \prod_{3}^{s} (1 - a_i)$
$\vdots$	$\vdots$
$p_0 + q_{s-1} + q_s$	$a_{s-1} a_s \prod_{1}^{s-2} (1 - a_i)$
$\vdots$	$\vdots$
$p_0 + \sum_{1}^{s} q_i$	$\prod_{1}^{s} a_i$

The product of $f_{\underline{Q}|N}(\underline{q}|\underline{n})$ and $P(Z = z|A)$ gives the joint distribution of the parameter $\underline{Q}$ and the achieved reliability Z, given the results of the N items tested. If we now integrate this over the ranges of the variables in $\underline{q}$, we are left with the marginal of Z, given the results of the N tests.

As we have seen, Z is equal to P_0 with probability $\prod_{i=1}^{s}(1 - a_i)$; then the marginal of Z, given A, is equal to the marginal posterior density of P_0 with this probability. Z is equal to $P_0 + Q_1$ with probability $a_1 \prod_{i=2}^{s}(1 - a_i)$; then the marginal of Z is equal to the marginal posterior density of $P_0 + Q_1$ with this probability. Proceeding in this manner through the various values of Z, we see that the marginal of Z consists of a weighted sum of the marginal posterior densities of P_0, $P_0 + Q_1$, $P_0 + Q_2$, ..., $P_0 + Q_1 + Q_2 + \dots + Q_s$, where the weights are simply the values of the probability function of Z, given A; thus, there are 2^s terms in this sum. It remains merely to derive the actual form of the posterior density of such variables as $P_0 + Q_1 + Q_2$. For simplicity let us explicitly derive the marginal posterior density of $P_0 + Q_1 + Q_2$; all of the densities of the other sums will have the same form.

We assume then that we have tested N items and observed n_0 successes, n_i failures in mode i, with $i = 1, 2, \dots, s$, and no failures in modes $s + 1, s + 2, \dots, K$. Using the prior given in section 2, Silva (and Earnest) shows that the posterior density of

$P_0, Q_1, \ldots, Q_K$ is

$$f_{\underline{Q}|N}(\underline{q}|N) = G_1\, p_0^{n_0 + m_0 - 1} \prod_{i=1}^{s} q_i^{n_i + m_i - 1} \prod_{j=s+1}^{K} q_j^{m_i - 1}$$

where

$$G_1 = \frac{\Gamma(N + m)}{\prod_{i=1}^{s} \Gamma(n_i + m_i) \prod_{j=s+1}^{K} \Gamma(m_j)} ,$$

with $p_0 \geq 0$, $q_i \geq 0$, for all i, and $p_0 + \sum_{i=1}^{K} q_i = 1$. To derive the marginal of P_0, Q_1, and Q_2 we need to integrate out $Q_3, \ldots, Q_K$ in the above density. By a straightforward extension of results given by Earnest, this is

$$f_{P_0, Q_1, Q_2|N}(p_0, q_1, q_2|N) = G_2\, p_0^{n_0 + m_0 - 1}$$

$$\cdot\ q_1^{n_1 + m_1 - 1}\ q_2^{n_2 + m_2 - 1}$$

$$\cdot\ (1 - p_0 - q_1 - q_2)^{N + m - (n_0 + n_1 + n_2) - (m_0 + m_1 + m_2) - 1}$$

where

$$G_2 = \frac{\Gamma(N + m)}{\prod_{i=0}^{2} \Gamma(n_i + m_i)\ \Gamma(N + m - n_0 - n_1 - n_2 - m_0 - m_1 - m_2)}$$

and $p_0 \geq 0$, $q_1 \geq 0$, $q_2 \geq 0$, $p_0 + q_1 + q_2 \leq 1$. Letting $w = p_0 + q_1 + q_2$ and integrating q_1 and q_2 over their ranges yields, as the marginal for $W = P_0 + Q_1 + Q_2$,

$$f_{W|N}(w|N) = G_3 \, w^{\sum_0^2 (n_i + m_i) - 1} (1 - w)^{N + m - \sum_0^2 (n_i + m_i) - 1} \quad \text{for } 0 < w < 1$$

$$= 0, \text{ otherwise,}$$

where

$$G_3 = \frac{\Gamma(N + m)}{\Gamma\left(\sum_0^2 (n_i + m_i)\right) \Gamma\left(N + m - \sum_0^2 (n_i + m_i)\right)} .$$

Thus, $W = P_0 + Q_1 + Q_2$ is a beta random variable with parameters $\sum_0^2 (n_i + m_i)$, $N + m - \sum_0^2 (n_i + m_i)$. It can be seen that P_0 plus the sum of any of the Q_i's will be a beta random variable; the effect of adding additional Q_i's to P_0 is simply to transfer the corresponding $(n_i + m_i)$'s from the second parameter to the first. The sum of the two parameters in the density function of P_0 plus any of the Q_i's is a constant and always equal to $N + m$ ($m = \sum_{i=0}^{K} m_i$, and N is equal to the number of items tested).

To see the significance of changes in the relative values of the two parameters of a beta random variable, let us note some general results. If X is a beta random variable with parameters a and b

(thus, the density for X is

$$B(x; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\, x^{a-1}(1-x)^{b-1} \qquad 0 < x < 1$$

and a is the first, b the second parameter), it is easily shown that the mean of X is $\frac{a}{a+b}$ and the mode of X (corresponding to the maximum value of $B(x; a, b)$) is $\frac{a-1}{a+b-2}$. Thus, if we consider a beta random variable for which $a + b$ is fixed, but a is free to vary, the larger the first parameter gets the further to the right both the center of gravity (mean) and the maximum (mode) of $B(x; a, b)$ will be; thus, more probability is concentrated at larger values of X. Thus, the density function of $P_0 + Q_1 + Q_2$ has more weight for higher values than does $P_0 + Q_1$.

Formally, we can explicitly write the density function for Z, given A, as

$$f_Z(z \mid A) = \sum_{x_1=0}^{1} \sum_{x_2=0}^{1} \cdots \sum_{x_s=0}^{1} \prod_{i=1}^{s} a_i^{x_i}(1-a_i)^{1-x_i}$$

$$\cdot\, B(z;\, n_0 + m_0 + \sum_{j=1}^{s} x_j(n_j + m_j),$$

$$N + n_0 + m - m_0 - \sum_{j=1}^{s} x_j(n_j + m_j))$$

where $B(z; a, b)$ is the density function of a beta random variable

with parameters a and b . Notice that the latter terms in this sum (the ones with the most x_i's equal to one) will be the ones with most probability at the highest values. The mean value of Z then is easily shown to be

$$\mu_{Z|A} = \frac{1}{N+m} \sum_{x_1=0}^{1} \cdots \sum_{x_s=0}^{1} \prod_{i=1}^{s} a_i^{x_i} (1-a_i)^{1-x_i} \left[n_0 + m_0 + \sum_{j=1}^{s} x_j (n_j + m_j) \right] .$$

Notice that the mean value of Z , given A , will increase as n_i increases, $i = 1, 2, \ldots, s$, which seems rather curious, since the number of times a mode occurs during the N trials does not affect the probability it will be corrected by the design change.

The variance of achieved reliability, Z , given the results of the N tests, can be computed in particular cases rather easily; however, a general expression of the variance of Z in terms of the variances of $P_0, Q_1, \ldots, Q_K$ would be too cumbersome to use. It is easily shown that if X is a beta random variable with parameters a and b , then

$$E(X^2) = \frac{a(a+1)}{(a+b)(a+b+1)} ;$$

accordingly,

$$E[Z^2|A] = \frac{1}{(N+m)(N+m+1)} \sum_{x_1=0}^{1} \cdots \sum_{x_s=0}^{1} \prod_{i=1}^{s} a_i^{x_i} (1-a_i)^{1-x_i} C(x_i, \ldots, x_s)$$

where

$$C(x_1, x_2, \ldots, x_s) = \left[n_0 + m_0 + \sum_{i=1}^{s} x_i (n_i + m_i) \right]$$

$$\cdot \left[n_0 + m_0 + \sum_{j=1}^{s} x_j (n_j + m_j) + 1 \right] .$$

The variance of Z, given the results of the N tests, then is

$$\sigma^2_{Z|A} = E[Z^2|A] - [E(Z|A)]^2 ,$$

which is easily evaluated in any particular case.

Interval statements about Z are easily made. Clearly,

$$P(Z > b|A) = \int_b^1 f_{Z|A}(z|A|dz)$$

$$= \sum_{x_1=0}^{1} \cdots \sum_{x_s=0}^{1} \prod_{i=1}^{s} a_i^{x_i} (1 - a_i)^{1 - x_i}$$

$$\cdot \int_b^1 B(z; C, N + m - C) dz$$

where

$$C = n_0 + m_0 + \sum_{j=1}^{s} x_j (n_j + m_j) .$$

Thus, for any given b, it is a simple table lookup to evaluate the integrals involved in the sum which determine the value of the probability statements. If we want to fix the value of the probability statement, at .95 for example, we can use the fact that $P(Z > b \mid A)$ is a monotonically decreasing function of b to numerically zero in on the desired value of b (for the fixed probability).

The next section is devoted to a particular numerical example showing the determination of the prior, the marginal mean of Z, the marginal variance of Z, and the value b such that

$$P(Z > b \mid A) = .95 \quad .$$

4. A NUMERICAL EXAMPLE

Let us suppose that we have designed an item (a missile or component thereof, for example) which consists of four subcomponents connected in series. The (unknown) probabilities that these four will work correctly are denoted by P_1^*, P_2^*, P_3^*, P_4^*, respectively. The probability that one of the items works correctly then is $P_0 = P_1^* P_2^* P_3^* P_4^*$ (the initial reliability). The probability of observing a failure in mode i, i = 1, 2, 3, 4, is

$$Q_i = \frac{1 - P_i^*}{4 - \sum_{j=1}^{4} P_j^*} [1 - P_0] \quad .$$

Assume that the following table gives prior failure information on each of the four subcomponents.

Subcomponent Number i	Number Tested r_i	Number of Successes X_i (in r_i tests)	Proportion of Successes $P_i^* = \frac{X_i}{r_i}$	Estimated Variance $\hat{V}_i = \frac{\hat{P}_i^* (1 - \hat{P}_i^*)}{r_i}$
1	1,000	990	.99	$.99 \times 10^{-5}$
2	10,000	9,800	.98	$.196 \times 10^{-5}$
3	5,000	4,750	.95	$.95 \times 10^{-5}$
4	200	194	.97	$.1455 \times 10^{-3}$

Then, using the method suggested in section 2 to estimate the parameters of the prior for P_0, Q_1, Q_2, Q_3, Q_4, we first compute the estimated initial reliability and the estimated probabilities of occurrence of the four failure modes using

$$\hat{P}_0 = \prod_{j=1}^{4} \hat{P}_j^*$$

$$Q_i = \frac{1 - \hat{P}_i^*}{4 - \Sigma \hat{P}_j^*} [1 - P_0], \quad i = 1, 2, 3, 4 .$$

The values of these quantities are displayed in the following table.

$\hat{P}_0$	$\hat{Q}_1$	$\hat{Q}_2$	$\hat{Q}_3$	$\hat{Q}_4$
.8940	.0096	.0193	.0482	.0289

We next determine the value of m, using Silva's method, to be

$$m = \frac{\Sigma [\hat{Q}_i]^2 [1 - \hat{Q}_i]^2}{\Sigma \hat{V}_i [\hat{Q}_i] [1 - \hat{Q}_i]} - 1$$

$$= 23.76$$

and then compute m_1, m_2, m_3, m_4 by $m_i = m \hat{Q}_i$, $i = 1, 2, 3, 4$ (we define $m_0 = m \hat{P}_0 = m - \sum_{i=1}^{4} m_i$). This yields $m_1 = .228$, $m_2 = .459$, $m_3 = 1.145$, $m_4 = .687$, and $m_0 = 21.241$.

These values of m and the m_i's, then, completely specify the prior distribution of P_0, Q_1, Q_2, Q_3, Q_4.

Now, let us assume that $N = 15$ of these items are tested and that we observe $n_0 = 13$ successes and $n_3 = 1$ failure in mode 3 and $n_4 = 1$ failure in mode 4. The two failures are analyzed, the design is changed, and it is decided that mode 3 has been corrected with probability $a_3 = .9$ and that mode 4 has been corrected with probability $a_4 = .99$.

The marginal distribution of achieved reliability Z, given the results of the 15 tests, is

$$
\begin{aligned}
f_{Z|A}(z|A) &= .001\ B(z;\ 34.241,\ 4.519) \\
&+ .099\ B(z;\ 35.928,\ 2.832) \\
&+ .009\ B(z;\ 36.386,\ 2.374) \\
&+ .891\ B(z;\ 38.073,\ 0.687) \quad ,
\end{aligned}
$$

where $B(z;a,b)$ again is the density function of a beta random variable with parameters a and b. The mean value of Z is

$$
\begin{aligned}
\mu_{Z|A} &= \frac{1}{38.76}\left[(.001)(34.241) + (.099)(35.928)\right. \\
&\qquad \left. + (.009)(36.386) + (.891)(38.073)\right] \\
&= .976 \quad ;
\end{aligned}
$$

we also find

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$$

we also find

$$E[Z^2 \mid A] = \frac{1}{38.76\,(39.76)} [(.001)(1206.687) + (.099)(1326.749) + (.009)(1360.327) + (.891)(1487.626)]$$

$$= .9540 .$$

Then, the variance of Z is

$$\sigma^2_{Z \mid A} = .9540 - (.976)^2 = .001424 ,$$

and the standard deviation of Z is

$$\sigma_{Z \mid A} = \sqrt{.001424} = .038 .$$

Let us now illustrate the computation of probability statements about Z. First, the probability that Z exceeds $.95$ is

$$\int_{.95}^{1} f_{Z \mid A}(z \mid A)\, dz ,$$

which in turn is equal to the weighted sum of the integrals of the four densities which make up $f_Z(z \mid a)$. Using rough interpolation in Pearson's tables of the incomplete beta-function [4], we find

$$\int_{.95}^{1} B(z;\ 34.241,\ 4.519) = .069 ,$$

$$\int_{.95}^{1} B(z;\ 35.928,\ 2.832) = .372 ,$$

$$\int_{.95}^{1} B(z;\ 36.386,\ 2.374) = .453 ;$$

$$\int_{.95}^{1} B(z;\ 38.073,\ .687) = .918 ;$$

thus,

$$P(Z > .95 \mid A) = (.001)(.069) + (.099)(.372) + (.009)(.453) + (.891)(.918)$$

$$= .859 \ .$$

Suppose we wanted to find the number c such that

$$P(Z > c \mid A) = .95 \quad ;$$

since this probability is a nonincreasing function of c, we know that $c < .95$. We then try a value of c (smaller than $.95$) such that

$$P(Z > c^* \mid A) > .95 \quad ;$$

then this new value c^* and $c = .95$ bracket the desired value of c; and by trying intermediate values, we can finally find c such that

$$P(Z > c \mid A) = .95 \quad .$$

Specifically, again by rough interpolation in Pearson's tables, we find

$$P(Z > .9 \mid A) = .9666 \quad ,$$

and thus the desired value lies between $.9$ and $.95$. By trying further values between $.9$ and $.95$, we find that

$$P(Z > .91 \mid A) = .9587$$

$$P(Z > .92 \mid A) = .9442 \quad ,$$

and thus the desired value lies between $.91$ and $.92$. By further interpolation in this interval, we could get as close as we like to the value having probability $.95$.

5. CONCLUSIONS, RECOMMENDATIONS

A Bayesian argument has been put forth leading to a completely known distribution for the achieved reliability following corrective action. This distribution can be used to give either point or interval statements about the final reliability. The resulting final distribution is dependent upon the type of prior assumed for the parameters. It is felt by the author that a very reasonable prior distribution has been assumed for the parameters, and arguments have been given supporting this feeling. A possible topic for further study, should this particular model prove quite useful, would be to study the manner in which the marginal distribution for Z changes as the assumed prior is changed; if the final distribution for Z is relatively stable for a wide variety of assumed priors, then very strong conclusions regarding Z can be reached, essentially independent of the assumed prior.

Rather than perform the sensitivity analysis just mentioned, the author would recommend a restructuring of the model to make it more realistic. There seem to be two basic weaknesses in the model assumed so far: the first of these is the fact that no provision is made in the model for the reliability possibly being decreased by the design change, which could definitely be the case in practical situations; the second defect is the assumption that all N tests are made simultaneously and that a single design change is made. In reality, tests of expensive pieces

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of equipment proceed sequentially; the first time a failure mode occurs, an attempt is made to remove it by a design change. Then, testing resumes on the new type of design until a failure mode occurs again, etc., at which point another design change is made. This practical sequential ordering of changes being made should be reflected in the model.

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A particular model is proposed for reliability growth and a prior distribution is assumed on the parameters. Various statements regarding the final reliability are then derived; a numerical example is included.

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